Comment on the quantum modes of the scalar field on AdS_{d+1} spacetime

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Abstract

The problem of the quantum modes of the scalar free field on anti-de Sitter backgrounds with an arbitrary number of space dimensions is considered. It is shown that this problem can be solved by using the same quantum numbers as those of the nonrelativistic oscillator and two parameters which give the energy quanta and respectively the ground state energy. This last one is known to be just the conformal dimension of the boundary field theory of the AdS/CFT conjecture.

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The recent interest in propagation of quantum scalar fields on anti-de Sitter (AdS) spacetime is due to the discovery of the AdS/Conformal field theory-correspondence [1]. One of central points here is the relation between the field theory on the $(d+1)$-dimensional AdS (AdS_{d+1}) spacetime and the conformal field theory on its d-dimensional Minkowski-like boundary (M_d). There are serious arguments that the local operators of the conformal field theory on M_d correspond to the quantum modes of the scalar field on AdS_{d+1} [2]. Actually, for $d = 3$ [3, 4] as well as for any d [5] it is proved that the conformal dimension in boundary field theory is equal with the ground state energy on AdS_{d+1} [2]. Moreover, it is known that the energy spectrum is discrete and equidistant [5] its quanta wavelength being just the hyperboloid radius of AdS_{d+1}.

In these conditions the scalar field on AdS_{d+1} can be seen as the relativistic correspondent of the nonrelativistic harmonic oscillator in d space dimensions. This means that the radial motion of the relativistic field may be governed by the same quantum numbers as that of the nonrelativistic oscillator, namely the radial and the angular quantum numbers. In the case of $d = 3$ we know that this is true [3, 6] but for $d > 3$ the definition and the role of the angular quantum number are not completely elucidated.

This is the motive why we would like to comment on this subject. Our aim is to present here the form of the normalized wave functions of the scalar field on AdS_{d+1} in terms of the above mentioned quantum numbers and to establish the formula of the degree of degeneracy of the energy levels. This problem is not complicated but in arbitrary dimensions some interesting technical details are worth reviewing. For this reason, we start with the separation of spherical variables in the Klein-Gordon equation on any central chart with d space coordinates and then turn to the AdS_{d+1} problem.

Let us consider a static local chart of a $(d + 1)$-dimensional spacetime where the coordinates x^μ, $\mu = 0, 1, ..., d$ are the time, $x^0 = t$, and the Cartesian space coordinates $\mathbf{x} \equiv (x^1, x^2, ..., x^d)$, while the signature of the metric tensor $g_{\mu\nu}(\mathbf{x})$ is $(+, -, -, ..., -)$. The charged scalar quantum field, $\phi \neq \phi^+$, of mass M, minimally coupled with the gravitational field, obeys the Klein-Gordon equation

$$\frac{1}{\sqrt{g}}\partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \phi\right) + M^2 \phi = 0\,, \quad g = |\det(g_{\mu\nu})|\,, \tag{1}$$

written in natural units with $\hbar = c = 1$. Since the chart is static there is a conserved energy, E, and, consequently, Eq.(1) has particular solutions (of positive and negative frequency) of the form

$$\phi_E^{(+)}(t, \mathbf{x}) = \frac{1}{\sqrt{2E}} e^{-iEt} U_E(\mathbf{x}), \quad \phi^{(-)} = (\phi^{(+)})^*, \tag{2}$$

which give us the one-particle quantum modes. These solutions may be even square integrable functions or tempered distributions on the domain D of the space coordinates of the local chart. In both cases they must be orhonormal (in usual or generalized sense) with respect to the relativistic scalar product [7]

$$\langle \phi_E, \phi_{E'} \rangle = i \int_D d^d x \, \sqrt{g} g^{00} \, \phi_E{}^* \overleftrightarrow{\partial_0} \, \phi_{E'} = \int_D d^d x \, \sqrt{g} g^{00} \, U_E^* U_{E'}, \tag{3}$$

which reduces to that of the static wave functions $U_E(\mathbf{x})$.

In the following, we take into account only static central backgrounds that have static and spherically symmetric local charts where the line element is invariant under global rotations, $R \in SO(d)$, of the Cartesian coordinates, $\mathbf{x} \to \mathbf{x}' = R\mathbf{x}$. In these charts the metric is diagonal in generalized spherical coordinates, $r, \theta_1, ..., \theta_{d-1}$, defined as [8]

$$\begin{aligned} x^1 &= r \sin\theta_1 ... \sin\theta_{d-1}, \\ x^2 &= r \sin\theta_1 ... \cos\theta_{d-1}, \\ &\vdots \\ x^d &= r \cos\theta_1 . \end{aligned} \tag{4}$$

such that the radial coordinate $r = |\mathbf{x}|$ be just the Euclidian norm of $\mathbf{x}$. These coordinates cover the domain $D = D_r \times S^{d-1}$, i.e. $r \in D_r$ while $\mathbf{x}/r$ is on the sphere S^{d-1}. In general, the line element,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00}(r) dt^2 + g_{rr}(r) dr^2 + g_{\theta\theta}(r)\, d\theta^2 \,, \tag{5}$$

with the angular part

$$d\theta^2 = d\theta_1{}^2 + \sin^2\theta_1 d\theta_2{}^2 ... + \sin^2\theta_1 \, \sin^2\theta_2 ... \sin^2\theta_{d-2} d\theta_{d-1}{}^2 \tag{6}$$

depends on three arbitrary functions of r, g_{00}, g_{rr} and $g_{\theta\theta} \equiv g_{\theta_1\theta_1}$. Hereby we find that

$$\sqrt{g(\mathbf{x})} = \sqrt{\hat{g}(r)}\, (\sin\theta_1)^{d-2} (\sin\theta_2)^{d-3} ... \sin\theta_{d-2} \,, \tag{7}$$

where

$$\hat{g} = |g_{00} g_{rr} {g_{\theta\theta}}^{d-1}|. \tag{8}$$

Furthermore, with the help of the new function

$$\rho = \hat{g}^{1/4} |g_{rr}|^{-1/2} \,, \tag{9}$$

we obtain the static Klein-Gordon equation

$$\left[\partial_r^2 + 2\partial_r(\ln\rho)\partial_r + g_{rr} g^{\theta\theta} \Delta_S + g_{rr} M^2\right] U_E = g_{rr} g^{00} E^2 U_E \tag{10}$$

which concentrates all the angular derivatives in the angular Laplace operator Δ_S [8]. We observe that this equation becomes an energy squared eigenvalue

problem in a *special* holonomic frame defined such that $g_{rr} = -g_{00}$. This condition can be achieved anytime with the help of an appropriate transformation of the radial coordinate of the central chart.

The spherical variables of Eq.(10) can be separated by using generalized spherical harmonics, $Y^{d-1}_{l\,(\lambda)}(\mathbf{x}/r)$. These are eigenfunction of the angular Lalpace operator [8],

$$-\,\Delta_S Y^{d-1}_{l\,(\lambda)}(\mathbf{x}/r) = l(l+d-2)\, Y^{d-1}_{l\,(\lambda)}(\mathbf{x}/r), \tag{11}$$

corresponding to eigenvalues depending on the *angular* quantum number l which take the values $0, 1, 2, ...$ [8]. The notation (λ) stands for a collection of quantum numbers giving the multiplicity of these eigenvalues [8],

$$\gamma_l = (2l+d-2)\frac{(l+d-3)!}{l!\,(d-2)!}. \tag{12}$$

Starting with particular solutions of the form

$$U_{E,l(\lambda)}(\mathbf{x}) = \frac{1}{\rho(r)} R_{E,l}(r)\, Y^{d-1}_{l\,(\lambda)}(\mathbf{x}/r), \tag{13}$$

after a few manipulation, we find the radial equation in a special frame

$$\left(-\frac{d^2}{dr^2} + g_{rr}g^{\theta\theta} l(l+d-2) - g_{rr}M^2 + \frac{1}{\rho}\frac{d^2\rho}{dr^2}\right) R_{E,l} = E^2 R_{E,l} \tag{14}$$

and the radial scalar product

$$\langle R_{E,l}, R_{E',l}\rangle = \int_{D_r} dr\, R_{E,l}(r)^* R_{E',l}(r)\,. \tag{15}$$

Here we have considered that the generalized spherical harmonics are normalysed to unity with respect to their own angular scalar product defined on the sphere S^{d-1}. Thus we obtain an independent radial problem in a special frame where the radial scalar product is of the simplest form. This is the starting point for finding analytical solutions of the Klein-Gordon equation on concrete central backgrounds.

Let us consider now the problem of the scalar field on AdS_{d+1}. This is a hyperboloid in the $(d+2)$-dimensional flat spacetime of coordinates Z^{-1}, Z^0, Z^1, ..., Z^d and metric

$$\eta_{AB} = \mathrm{diag}(1, 1, -1, ..., -1)\,, \quad A,\, B = -1, 0, 1, ...d\,. \tag{16}$$

The hyperboloid equation reads

$$(Z^{-1})^2 + (Z^0)^2 - (Z^1)^2 ... - (Z^d)^2 = R^2 \tag{17}$$

where $R = 1/\omega$ is its radius. In a special frame the coordinates (4) satisfy

$$\begin{aligned} Z^{-1} &= \frac{1}{\omega} \sec \omega r \cos \omega t \\ Z^0 &= \frac{1}{\omega} \sec \omega r \sin \omega t \qquad (18) \\ \mathbf{Z} &= \frac{1}{\omega} \tan \omega r \, \frac{\mathbf{x}}{r}, \end{aligned}$$

$$\tag{19}$$

giving the line element [3, 5]

$$ds^2 = \eta_{AB} dZ^A dZ^B = \sec^2 \omega r \left(dt^2 - dr^2 - \frac{1}{\omega^2} \sin^2 \omega r \, d\theta^2 \right) . \tag{20}$$

on the radial domain $D_r = [0, \, \pi/2\omega)$. From Eqs.(18) it results that the time of AdS_{d+1} must satisfy $t \in [-\pi/\omega, \pi/\omega)$. We remind that $t \in (-\infty, \infty)$ defines the universal covering spacetime of AdS_{d+1} ($CAdS_{d+1}$) [3].

Now from (20) we identify the components of the metric tensor and we find

$$\rho(r) = \left(\frac{1}{\omega} \tan \omega r \right)^{\frac{d-1}{2}} . \tag{21}$$

With these ingredients and by using the notations $\epsilon = E/\omega$ and $\mu = M/\omega$ (i.e. $\epsilon = E/\hbar\omega$ and $\mu = Mc^2/\hbar\omega$ in usual units), we obtain the radial equation

$$\left[-\frac{1}{\omega^2} \frac{d^2}{dr^2} + \frac{2s(2s-1)}{\sin^2 \omega r} + \frac{2p(2p-1)}{\cos^2 \omega r} \right] R_{E,l} = \epsilon^2 R_{E,l} \tag{22}$$

where

$$2s(2s-1) = \left(l + \frac{d}{2} - 1 \right)^2 - \frac{1}{4} , \quad 2p(2p-1) = \mu^2 + \frac{d^2-1}{4} . \tag{23}$$

It is well-known that the solutions of Eq.(22) can be expressed in terms of hypergeometric functions [9], up to normalization factors, as

$$R_{E,l}(r) \sim \sin^{2s} \omega r \cos^{2p} \omega r F \left(s + p - \frac{\epsilon}{2}, s + p + \frac{\epsilon}{2}, 2s + \frac{1}{2}, \sin^2 \omega r \right) . \tag{24}$$

These radial functions can have good physical meaning only as polynomials selected by a suitable quantization condition. This is because the above hypergeometric functions are so strongly divergent for $\sin^2 \omega r \to 1$ that $R_{E,l}$ can not be interpreted as tempered distributions corresponding to a continuous energy spectrum. Therefore, we introduce the radial quantum number n_r [5] and impose

$$\epsilon = 2(n_r + s + p)\,, \quad n_r = 0, 1, 2, ... \tag{25}$$

In addition, we choose the positive solutions of Eqs.(23) in order to avoid singularities in $r = 0$ and $r = \pi/2\omega$. These are

$$2s = l + \frac{d-1}{2}\,, \quad 2p = k - \frac{d-1}{2}\,, \tag{26}$$

where we have denoted by

$$k = \sqrt{\mu^2 + \frac{d^2}{4}} + \frac{d}{2} \tag{27}$$

the conformal dimension of the field theory on M_d [2]. We note that (25) is the quantization condition on $CAdS_{d+1}$ while the AdS_{d+1} one requires, in addition, k to be an integer number too [3].

The last step is to define the *main* quantum number, $n = 2n_r + l$, which take the values, $0, 1, 2, ...$, giving the energy levels

$$E_n = \omega(k + n) \tag{28}$$

If n is even then $l = 0, 2, 4, ..., n$ while for odd n we have $l = 1, 3, 5, ..., n$. In both cases we can demonstrate that the degree of degeneracy of the level E_n is

$$\gamma_n = \sum_l \gamma_l = \frac{(n+d-1)!}{n!\,(d-1)!}\,. \tag{29}$$

Now it is a simple exercise to express (24) in terms of Jacobi polynomials and to normalize them to unity with respect to (15). Then by using (21) and (13) we restore the final form of the solutions (2),

$$\phi^{(+)}_{n,l(\lambda)}(t, \mathbf{x}) = N_{n,l} \sin^l \omega r \cos^k \omega r P_{n_r}^{(l+\frac{d}{2}-1,\, k-\frac{d}{2})}(\cos 2\omega r) Y^{d-1}_{l(\lambda)}(\mathbf{x}/r)\, e^{-iE_n t}, \tag{30}$$

where

$$N_{n,l} = \omega^{\frac{d-1}{2}} \left[\frac{n_r!\,\Gamma(n_r+k+l)}{\Gamma(n_r+l+\frac{d}{2})\Gamma(n_r+k+1-\frac{d}{2})} \right]^{\frac{1}{2}} . \tag{31}$$

Thus we have shown that the problem of the one-particle quantum modes of the scalar field on $CAdS_{d+1}$ can be solved by using the quantum numbers $n/n_r, l, (\lambda)$ and parameters with precise physical interpretation, i.e. the frequency of the energy quanta, $\omega = 1/R$, and the conformal dimension, k, which gives the ground state energy. We must specify that these results coincide with those of Refs.[3, 4, 6] for $d = 3$ but in the general case of any d these are similar (up to notations) to those of Ref.[5] only for $l = 0$ while for $l \neq 0$ there are some differences.

Finally we note that the parameter k we use instead of M could play an important role in the supersymmetry and shape invariance of the radial problem as well as in the structure of the dynamic algebra. The argument is that the radial problems for arbitrary d are of the same nature as that with $d = 1$ for which we have recently shown that k determines the shape of the relativistic potential and, in addition, represents the minimal weight of the irreducible representation of its $so(1,2)$ dynamic algebra [10].

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